Generalized hyperpolygons and applications

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This short note gives an overview of generalized hyperpolygons introduced by the author and Steven Rayan in [9], and some directions of interest related to them with particular attention given to their relation to Higgs bundles and their interpretation within F-theory [4]. The manuscript has been prepared for the ZAG volume, is based on the authors' talk in the ZAG seminar series during the COVID-19 pandemic, and contains no new results.

Generalized hyperpolygons via A-type quiver varieties. Hyperpolygons were first studied by Konno [7] as a natural hyperkähler extension of the usual polygon space. These spaces are closely related to Higgs bundles, as was first seen for genus 0 by Godinho-Mandini [3] for rank 2, and later extended to any rank with minimal flags by Fisher-Rayan [2]. More recently, with Steve Rayan we have considered the case of Higgs bundles on surfaces of arbitrary genus, and of hyperpolygons (and generalized ones) with any type of flags [9, 10, 11].

Recall that given an equioriented A-type quiver of length m with vertices labelled by an increasing sequence of nonnegative numbers r_i, one may associate the vector space V and a group G acting freely on it by conjugation:

$$V = \bigoplus_{i=1}^{m-1} \mathrm{Hom}(\mathbb{C}^{r_i}, \mathbb{C}^{r_i - 1}), \text{ and } G = \prod_{i=1}^{m-1} \mathrm{GL}(r_i, \mathbb{C})$$

By restricting to the subspace of V on which G acts free and considering the quotient space, one obtains the quotient $V/\!\!/G$ which is a *partial flag variety* $\mathcal{F}_{r_1,\ldots,r_m}$. By considering the quaternionic vector space T^*V associated to the doubled quiver, rather than V, through the quotient one obtains the corresponding Nakajima quiver variety, which can be seen as a symplectic quotient $T^*V/\!\!/\!\!/G$ inheriting three quaternionically-commuting complex structures I, J, K from T^*V as well as a Riemannian metric [6]. The space of *generalized hyperpolygons* can be then defined through a similar procedure on the double quivers of in Figure 1 below.

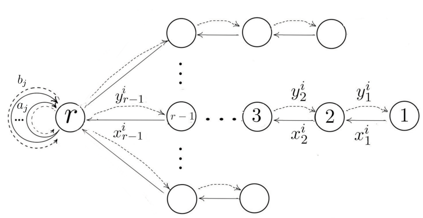

FIGURE 1. A come-shaped quiver leading to generalized hyperpolygons

From Hyperpolygons to Higgs bundles and back. The moduli space of generalized hyperpolygons builds upon the space of hyperpolygons [7], which appears as the Nakajima quiver variety of star-shaped quivers $\mathcal{Q}$ (fe.g., see [11]). We refer to the A-type quivers as the *arms* of the star-shaped quiver and by adding g loops to the central node of $\mathcal{Q}$ we get a *comet quiver*, an example of which is shown in Figure 1. The moduli space of generalized hyperpolygons can then be defined as the Nakajima quiver variety obtained via symplectic reduction through two moment maps [9], and gives an integrable system:

Theorem ([9]) *When each arm of the comet quiver is either complete or minimal, the moduli space $\mathcal{X}^g_{\underline{r^1},\ldots,\underline{r^n}}(\underline{\alpha})$ is an algebraically completely integrable Hamiltonian system of Gelfand-Tsetlin type with Hamiltonians depending only on the data $[x, y, a, b]$ of a representation.*

At the central node r the *hyperpolygon equations* coming from the two level sets of the moment maps are:

$$\text{(i)} \quad \sum_{i=1}^{n}(x^i_{m_i-1}(x^i_{m_i-1})^* - (y^i_{m_i-1})^* y^i_{m_i-1})_0 + \sum_{j=1}^{g}[a_j, a_j^*] + [b_j, b_j^*] = 0$$

$$\text{(I)} \quad \sum_{i=1}^{n}(x^i_{m_i-1} y^i_{m_i-1})_0 + \sum_{j=1}^{g}[a_j, b_j] = 0$$

These equations can be seen as discrete analogues of the *Hitchin equations* [5], which are equations on a smooth Hermitian bundle E over a Riemann surface X. Indeed, equation (i) is the analogue of the first Hitchin equation

$$F(A) + \phi \wedge \phi^* = 0$$

in which the failure of a unitary connection A on E to be flat is expressed in terms of a *Higgs field* $\phi : E \to E \otimes K$, where ϕ is linear in functions and K is the canonical line bundle of X. In particular, the failure of the connection to be flat is paralleled in the failure of the (ordinary) polygon to close. The way in which the Higgs field "flattens" the connection (we have $F(A + \phi + \phi^*) = 0$) is paralleled by the way in which the y and b data close the figure in $\mathfrak{su}(r)$. In a similar fashion, equation (I) is the analogue of the second Hitchin equation

$$d''_A \phi = 0$$

that makes ϕ holomorphic with respect to the holomorphic structure on E induced by A. Equation (I) can be thought of as "holomorphicity at infinity" for an associated Higgs bundle, which we describe now.

The way we associate a corresponding Higgs field to a (generalized) hyperpolygon is by considering a punctured surface. Let $D = \sum_i z_i$ be a positive divisor in the upper half plane which is tiled with punctured $4g$-gons. Then, one can use the x and y data to define the residues of the Higgs field which now is meromorphic at these points:

$$\phi(z) = \sum_{i=1}^{n} \frac{x_i y_i}{z - g_z(z_i)} dz,$$

for $g_z(z_i)$ chosen appropriately as a (quasi)periodic function [9]. This correspondence gives an embedding that respects the I structure but not the J and the K.

Hyperpolygons and dualities. Inspired by questions in F-theory [1, 4], one may allow comet-shaped quivers in which two or more edges e are permitted between two consecutive vertices along an arm (and therefore two or more corresponding arrows in the doubled quiver). Building upon the relation with Higgs bundles, we refer to such a quiver as a *wild comet* and describe classes in the resulting quiver variety as *wild hyperpolygons*, and examples of such quivers appear below in Figure 2. From this point of view, our generalized hyperpolygons would be *tame hyperpolygons* representing a *tame comet*.

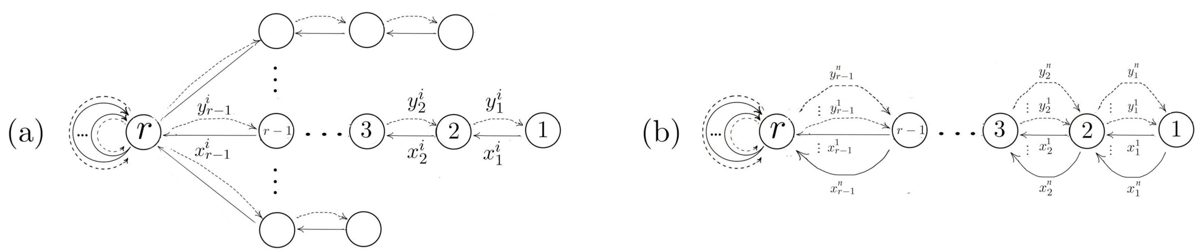

FIGURE 2. (a) An example of a tame comet; (b) an example of a wild comet, whose pole order at the marked point is given by the number of the same direction arrows between two consecutive nodes.

In the context of 6-dimensional F-theory compactifications by varying the complex structure moduli of a singular CY variety, the location of singularities in the discriminant locus of the system can be tuned to coincide. This tuning, when viewed from the intersecting brane models, should correspond to a parametric deformation of a Hitchin System in which the location of simple poles are tuned and forced to coincide into higher order poles [1, 4]. In particular this leads to the following conjecture.

Conjecture 1 ([4])**.** *There exists a morphism between the moduli spaces of parabolic Higgs bundles and that of wild Higgs bundles in the case of a parabolic Higgs bundle with n-simple poles and that of a wild Higgs bundle with a single higher order pole of order n.*

As mentioned in [9], the above conjecture can be thought of from the perspective of hyperpolygons in the following way. Consider the case where $\mathcal{X}^g_{\underline{r^1},\dots,\underline{r^n}}(\underline{\alpha})$ arises from a tame comet in which each flag is the same — for example, the complete and minimal comets. In such setting, we can consider a hyperpolygon $[x, y, a, b]$ as being a representation of a tame comet *or* of a wild comet with a single arm but with n-many x arrows and n-many y arrows connecting any two consecutive nodes. The wild comet comes about by identifying corresponding nodes of the arms. At the level of associated Higgs bundles, we are isolating a locus of wild Higgs bundles with an order-n pole at infinity that is constructed from a tame Higgs bundle with n-many order-1 poles, simply by rearranging the residues, leading to the following conjecture whose formalization has not been done yet.

Conjecture 2 ([9])**.** *One can think of the passage from an order-n pole to n-many order-*1 *poles by performing quiver mutations, and those in turn corresponding to degenerations of Painlevé equations.*

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